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Superfield approach to topological features of non-Abelian gauge theory

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Abstract

We discuss some of the key topological aspects of a (1 + 1)-dimensional (2D) self-interacting non-Abelian gauge theory (having no interaction with matter fields) in the framework of *chiral* superfield formalism. We provide the geometrical interpretation for the Lagrangian density, symmetric energy–momentum tensor, topological invariants, etc, by exploiting the *on-shell* nilpotent BRST and co-BRST symmetries that emerge after the application of (dual) horizontality conditions. We show that the above physically interesting quantities geometrically correspond to the translation of some local (but composite) *chiral* superfields along one of the two independent Grassmannian directions of a (2 + 2)-dimensional supermanifold. This translation is generated by the conserved and on-shell nilpotent (co-)BRST charges that are present in the theory.

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1. Introduction

The modern developments in the subject of topological field theories (TFTs) have encompassed in their ever-widening horizons a host of diverse and distinct areas of research in theoretical physics and mathematics. In this context, mention can be made of interesting topics such as Chern–Simon theories, topological string theories and matrix models, 2D topological gravity, Morse theory, Donaldson and Jones polynomials, etc (see, e.g., [1] and references therein for details). Without going into the subtleties and intricacies, TFTs can be broadly classified into two types. The Witten-type TFTs [2] are those where the Lagrangian density turns out to be the Becchi–Rouet–Stora–Tyutin (BRST) (anti-)commutator. The conserved and nilpotent BRST charge for such a class of TFTs generates a symmetry that is a combination of a topological shift symmetry and some types of local gauge symmetry. On the other hand, the Schwarz-type TFTs [3] are characterized by the existence of a conserved and nilpotent

BRST charge that generates only some local gauge type of symmetries for a Lagrangian density that *cannot* be completely expressed as the BRST (anti-)commutator (see, e.g., [1] for details). For both types of TFTs, there are no energy excitations in the physical sector of the theory because the energy–momentum tensor turns out to be a BRST (anti-) commutator and all the physical states (including the vacuum) of this theory are supposed to be invariant w.r.t. the conserved, nilpotent, metric-independent and Hermitian BRST charge Q_b (i.e. $Q_b|\mathrm{vac}\rangle = 0$, $Q_b|\mathrm{phys}\rangle = 0$).

Recently, in a set of papers [4–7], the free 2D Abelian and self-interacting non-Abelian gauge theories (without any kind of interaction with matter fields) have been shown to belong to a new class of TFTs because the Lagrangian density of the theory turns out to bear an appearance similar to the Witten-type theories but the local symmetries of the theory are that of Schwarz type. Furthermore, these non-interacting as well as self-interacting 2D theories [4–7], interacting 2D Abelian gauge theory (where there is an interaction between Dirac fields and 2D photon) [8, 9] and (3 + 1)-dimensional (4D) free Abelian 2-form gauge theory [10] have been shown to provide a set of tractable field theoretical models for the Hodge theory where the local, covariant and continuous symmetries of the Lagrangian density and corresponding conserved charges (as generators) are identified with all three de Rham cohomology operators of differential geometry[1]. The geometrical interpretation for these charges as the translation generators along some specific directions of a (2 + 2)-dimensional supermanifold has also been established for the 2D free and self-interacting (non-)Abelian gauge theories [15–17]. In a recent paper [18], some of the key features of the topological nature of a 2D free Abelian gauge theory have been captured in the superfield formulation [19–23] and their geometrical interpretations have been provided in the language of translations along some specific directions of the (2 + 2)-dimensional supermanifold. One of the central themes of our present paper is to extend our work on the free 2D Abelian gauge theory [18] to the *more general case* of self-interacting 2D non-Abelian gauge theory and provide the geometrical interpretation for some key topological properties associated with this theory in the framework of (geometrical) superfield formulation [19–23]. Such studies are important because they provide a geometrical origin for some of the key topological quantities of physical interest (e.g., Lagrangian density, symmetric energy–momentum tensor, topological invariants, etc) for the (non-)Abelian gauge theories. In particular, our results on the geometrical origin and interpretation for the Lagrangian density and corresponding symmetric energy–momentum tensor are completely novel in nature *vis-à-vis* the key results of [17, 20–22] where the geometrical interpretation for *only* the nilpotent (anti-)BRST charges [17, 20–22] (and (anti-) co-BRST charges [17]) has been provided. The key observations of our present paper are, however, similar to those of [18].

The self-interacting non-Abelian gauge theory (having no interaction with matter fields) is described by a singular Lagrangian density that happens to be endowed with first-class constraints in the language of Dirac's classification scheme [24, 25]. For the BRST quantization of such a theory, the original Lagrangian density is extended to include the gauge-fixing and Faddeev–Popov ghost terms so that the theory can maintain unitarity and 'quantum' gauge symmetry (i.e. nilpotent BRST symmetry) *together* at any arbitrary order of perturbation theory. The ensuing Lagrangian density that respects the *on-shell* nilpotent BRST symmetry, however, does not respect the corresponding on-shell nilpotent anti-BRST symmetry. To the best of our familiarity with the relevant literature, the on-shell nilpotent

[1] On a compact manifold without a boundary, the set of operators $(\mathrm{d}, \delta, \Delta)$ (with $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$, $\delta = \pm * \mathrm{d}*$, $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$) define the de Rham cohomological properties of the differential forms. They are called the exterior derivative, co-exterior derivative and Laplacian operator, respectively, and obey $\mathrm{d}^2 = \delta^2 = 0$, $[\Delta, \mathrm{d}] = [\Delta, \delta] = 0$, $\Delta = \{\mathrm{d}, \delta\} = (\mathrm{d} + \delta)^2$. Here $*$ stands for the Hodge duality operation [11–14].

version of the anti-BRST symmetry does not exist for the self-interacting and/or interacting non-Abelian gauge theory in any arbitrary spacetime dimension. This feature of the self-interacting non-Abelian gauge theory is drastically different from its Abelian counterpart where both the on-shell nilpotent (anti-)BRST symmetries are respected by one and the same Lagrangian density (see, e.g., [15, 16]). A possible explanation for this discrepancy has been provided in a recent paper [26] by resorting to the geometrical superfield approach to BRST formalism. In our present paper, we obtain the on-shell nilpotent version of the BRST and co-BRST symmetries for the 2D self-interacting non-Abelian gauge theory by exploiting the generalized versions of horizontality condition[2] w.r.t. the (super) cohomological operators $(\tilde{\mathrm{d}})\mathrm{d}$ (together with the Maurer–Cartan equation) and $(\tilde{\delta})\delta$ defined on the $(2+2)$-dimensional supermanifold. In this endeavour, the choice of the superfields to be chiral plays a very decisive role as the (dual) horizontality conditions (w.r.t. $(\tilde{\delta})\delta$ and $(\tilde{\mathrm{d}})\mathrm{d}$) lead to the derivation of the on-shell nilpotent (co-)BRST symmetries *only*. The off-shell version of these symmetries has already been obtained in [17] where the ideas of [20–22] on the (anti-)BRST symmetries have been expanded and new (anti-)co-BRST symmetries have been introduced and derived in the superfield formalism by exploiting the above (super) cohomological operators together with the imposition of (dual) horizontality conditions. In contrast to the choice of chiral superfields for the derivation of the on-shell nilpotent (co-) BRST symmetries, the off-shell nilpotent (anti-) BRST and (anti-) co-BRST symmetries have been derived by taking into account the most general superfield expansion along the θ-, $\bar{\theta}$- and $\theta\bar{\theta}$-directions of the $(2+2)$-dimensional supermanifold [17].

In our present discussion, we concentrate only on the on-shell version of nilpotent BRST and co-BRST symmetries (avoiding any discussion about the anti-BRST and anti-co-BRST symmetries) because the basic Lagrangian density (see, e.g., (2.1)) respects only these symmetries. The derivations of these symmetries and corresponding generators are good enough to shed some light on the topological nature of the 2D self-interacting non-Abelian gauge theory. In fact, the topological nature of this theory is encoded in the form of the Lagrangian density and the symmetric energy–momentum tensor which can be thought of as the translation of some local (but composite) chiral superfields along one of the two Grassmannian directions of the supermanifold. This translation is generated by the on-shell nilpotent BRST and co-BRST charges which turn out to geometrically correspond to the translation generators along one of the two Grassmannian directions of the $(2+2)$-dimensional supermanifold. In mathematical terms, the Lagrangian density and symmetric energy–momentum tensor for the present theory turn out to be the total derivative of some local (but composite) chiral superfields w.r.t. one of the two Grassmannian variables (cf (5.1) and (5.5)). In these derivations, the chiral superfield expansions are taken to be those that are obtained after the application of (dual) horizontality conditions. Thus, the (super) cohomological operators $(\tilde{\delta})\delta$ and $(\tilde{\mathrm{d}})\mathrm{d}$ play very important and pivotal roles for our present discussions through the (dual) horizontality restrictions.

Our present investigation is essential primarily on three counts. First, to the best of our knowledge, the full potential of (super) co-exterior derivatives $(\tilde{\delta})\delta$ has not yet been thoroughly exploited in the context of the superfield approach to the BRST formalism except in some of our recent works [15–18]. Thus, besides whatever has been achieved and understood in [15–18, 26], it is important to explore the utility of these (super) cohomological operators in their diverse, distinct and multiple forms. Second, our present paper explains the reason behind the existence of on-shell nilpotent (co-)BRST symmetries for the Lagrangian density (cf (2.1)) that does not respect on-shell nilpotent anti-BRST and anti-co-BRST symmetries.

[2] This condition is referred to as the 'soul-flatness' condition by Nakanishi and Ojima [27].

In fact, the choice of the chiral superfields (along with the idea of (dual) horizontality conditions) plays an important role in proving the existence of on-shell nilpotent (co-)BRST symmetries. The choice of the anti-chiral superfields does not lead to the derivation of anti-BRST and anti-co-BRST symmetries for this theory as explained in our recent work [26]. Finally, the geometrical understanding of the Lagrangian density and the symmetric energy–momentum tensor for the present theory *might* turn out to be useful in understanding the topological 2D gravity and topological string theories where a non-trivial metric is chosen for the theoretical discussions of such kinds of gauge theories in the background of curved spacetime.

The contents of our present paper are organized as follows. In section 2, we set up the notation and briefly recapitulate the bare essentials of the BRST and co-BRST symmetries for the 2D self-interacting non-Abelian gauge theory in the Lagrangian formulation. Sections 3 and 4 are devoted to the derivation of the above on-shell nilpotent symmetries in the framework of superfield formalism. In section 5, we discuss topological aspects and provide their geometrical interpretations in the language of translations on the (2 + 2)-dimensional supermanifold. Finally, in section 6, we make some concluding remarks and point out a few directions that can be pursued later.

2. BRST and co-BRST symmetries: Lagrangian formulation

Let us begin with the BRST-invariant Lagrangian density $\mathcal{L}_B$ for the self-interacting (1 + 1)-dimensional[3] non-Abelian gauge theory in the Feynman gauge [27–30],

$$\begin{aligned}\mathcal{L}_B &= -\tfrac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu A^\mu)\cdot(\partial_\rho A^\rho) - \mathrm{i}\partial_\mu\bar{C}\cdot D^\mu C\\ &\equiv \tfrac{1}{2}E\cdot E - \tfrac{1}{2}(\partial_\mu A^\mu)\cdot(\partial_\rho A^\rho) - \mathrm{i}\partial_\mu\bar{C}\cdot D^\mu C\end{aligned} \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \times A_\nu$ is the field strength tensor derived from the connection 1-form $A = \mathrm{d}x^\mu A^a_\mu T^a$ (with $A_\mu = A^a_\mu T^a$ as the vector potential) by the application of the Maurer–Cartan equation $F = \mathrm{d}A + A \wedge A$ where $\mathrm{d} = \mathrm{d}x^\mu\partial_\mu$ is the exterior derivative (with $\mathrm{d}^2 = 0$) and the 2-form $F = \frac{1}{2}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)F^a_{\mu\nu}T^a$. In the 2D spacetime, only the electric field component ($F_{01} = E = E^a T^a$) of the field strength tensor $F_{\mu\nu}$ exists. Here T^a form the compact Lie algebra: $[T^a, T^b] = f^{abc}T^c$ with structure constants f^{abc} which can be chosen to be totally antisymmetric in a, b, c (see, e.g., [30] for details). The gauge-fixing term is derived as $\delta A = (\partial_\rho A^{\rho a}T^a)$ where $\delta = -*\mathrm{d}*$ (with $\delta^2 = 0$) is the co-exterior derivative and $*$ is the Hodge duality operation. The (anti-)commuting ($C^a\bar{C}^b + \bar{C}^b C^a = 0$, $(C^a)^2 = (\bar{C}^a)^2 = 0$) (anti-)ghost fields ($\bar{C}^a$) C^a are required in the BRST-invariant theory to maintain unitarity and 'quantum' gauge (i.e. BRST) invariance together at any arbitrary order of perturbative calculations. In fact, these (anti-)ghost fields (which are not matter fields) interact with the self-interacting non-Abelian gauge fields $\left(A_\mu = A^a_\mu T^a\right)$ only in the loop diagrams of perturbation theory (see, e.g., [31] for details). The above Lagrangian density (2.1) respects ($s_b\mathcal{L}_B = -\partial_\mu[(\partial_\rho A^\rho)\cdot D^\mu C]$, $s_d\mathcal{L}_B = \partial_\mu[E\cdot\partial^\mu\bar{C}]$) the following on-shell ($\partial_\mu D^\mu C = D_\mu\partial^\mu\bar{C} = 0$) nilpotent $\left(s_b^2 = 0, s_d^2 = 0\right)$ BRST (s_b) and dual(co)-BRST

[3] We adopt here the conventions and notation such that the 2D flat Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1)$ and $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_1)^2$, $F_{01} = F^{10} = -\varepsilon^{\mu\nu}(\partial_\mu A_\nu + \frac{1}{2}A_\mu \times A_\nu) = E = \partial_0 A_1 - \partial_1 A_0 + A_0 \times A_1$, $\varepsilon_{01} = \varepsilon^{10} = +1$, $D_\mu C = \partial_\mu C + A_\mu \times C$, $\alpha\cdot\beta = \alpha^a\beta^a$, $(\alpha\times\beta)^a = f^{abc}\alpha^b\beta^c$ for non-null vectors α and β in the group space. Here Greek indices $\mu, \nu \ldots = 0, 1$ correspond to the spacetime directions on the 2D manifold and Latin indices $a, b, c \ldots = 1, 2, 3 \ldots$ stand for the Lie group 'colour' values.

(s_d) symmetry[4] transformations [5, 6, 17, 26]:

$$
\begin{array}{llll}
s_b A_\mu = D_\mu C & s_b C = -\frac{1}{2} C \times C & s_b \bar{C} = -\mathrm{i}(\partial_\mu A^\mu) & s_b E = E \times C \\
s_d A_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C} & s_d \bar{C} = 0 & s_d C = -\mathrm{i}E & s_d E = D_\mu \partial^\mu \bar{C}.
\end{array}
\tag{2.2}
$$

The above continuous symmetries, according to Noether's theorem, lead to the following expressions for the conserved and on-shell nilpotent (co-)BRST charges (Q_d) Q_b [5, 6],

$$
\begin{aligned}
Q_b &= \int \mathrm{d}x \left[\partial_0(\partial_\rho A^\rho)\cdot C - (\partial_\rho A^\rho)\cdot D_0 C + \frac{\mathrm{i}}{2}\dot{\bar{C}}\cdot C \times C\right] \\
Q_d &= \int \mathrm{d}x [E \cdot \dot{\bar{C}} - D_0 E \cdot \bar{C} - \mathrm{i}\bar{C}\cdot \partial_1 \bar{C} \times C]
\end{aligned}
\tag{2.3}
$$

which turn out to be the generator for the transformations (2.2). This latter statement can be succinctly expressed in the mathematical form (for the generic field $\Psi = \Psi^a T^a$) as

$$
s_r \Psi = -\mathrm{i}[\Psi, Q_r]_\pm \qquad r = b, d \tag{2.4}
$$

where brackets $[\,,\,]_\pm$ stand for the (anti-)commutators for any arbitrary generic field Ψ $(\equiv A_\mu, C, \bar{C})$ being (fermionic) bosonic in nature. Left to itself, the Lagrangian density (2.1) does *not* respect any anti-BRST and anti-co-BRST symmetries. These symmetries can be brought in, however, by modifying (2.1) to incorporate a specific set of auxiliary fields (see, e.g., [5, 6]). It is interesting to note that only the *off-shell nilpotent* version of these symmetries exists for the modified Lagrangian density (see, e.g., [27–30] for details). With the help of equations (2.3) and (2.4), the Lagrangian density (2.1) can be expressed, modulo some total derivatives, as the sum of on-shell nilpotent BRST and co-BRST anti-commutators:

$$
\mathcal{L}_B = \left\{Q_d, \frac{1}{2}E\cdot C\right\} - \left\{Q_b, \frac{1}{2}(\partial_\rho A^\rho)\cdot \bar{C}\right\} \equiv s_d\left[\frac{\mathrm{i}}{2}E\cdot C\right] - s_b\left[\frac{\mathrm{i}}{2}(\partial_\rho A^\rho)\cdot\bar{C}\right]. \tag{2.5}
$$

The appearance of the above Lagrangian density is that of Witten-type TFTs when the physical states (and vacuum) of the theory are supposed to be annihilated by Q_b and Q_d. Such a situation does arise if we invoke the harmonic state of the Hodge decomposed state to correspond to the *physical state* in the total quantum Hilbert space [5–7]. The expression for the symmetric energy–momentum tensor $T^{(s)}_{\mu\nu}$ for the Lagrangian density $(\mathcal{L}_B)$ is

$$
\begin{aligned}
T^{(s)}_{\mu\nu} = -\frac{1}{2}(\partial_\rho A^\rho)\cdot(\partial_\mu A_\nu + \partial_\nu A_\mu) &- \frac{1}{2}E\cdot(\varepsilon_{\nu\rho}\partial_\mu A^\rho + \varepsilon_{\mu\rho}\partial_\nu A^\rho) \\
&- \frac{\mathrm{i}}{2}(\partial_\mu \bar{C})\cdot(D_\nu C + \partial_\nu C) - \frac{\mathrm{i}}{2}(\partial_\nu\bar{C})\cdot(D_\mu C + \partial_\mu C) - \eta_{\mu\nu}\mathcal{L}_B.
\end{aligned}
\tag{2.6}
$$

This also turns out, modulo some total derivatives, to be the sum of BRST and co-BRST anti-commutators as given below [6]:

$$
\begin{aligned}
T^{(s)}_{\mu\nu} &= \left\{Q_b, L^{(1)}_{\mu\nu}\right\} + \left\{Q_d, L^{(2)}_{\mu\nu}\right\} \equiv s_b\left(\mathrm{i}L^{(1)}_{\mu\nu}\right) + s_d\left(\mathrm{i}L^{(2)}_{\mu\nu}\right) \\
L^{(1)}_{\mu\nu} &= \tfrac{1}{2}[\partial_\mu\bar{C}\cdot A_\nu + \partial_\nu\bar{C}\cdot A_\mu + \eta_{\mu\nu}(\partial_\rho A^\rho)\cdot\bar{C}] \\
L^{(2)}_{\mu\nu} &= \tfrac{1}{2}[\partial_\mu C\cdot\varepsilon_{\nu\rho}A^\rho + \partial_\nu C\cdot\varepsilon_{\mu\rho}A^\rho - \eta_{\mu\nu}E\cdot C].
\end{aligned}
\tag{2.7}
$$

The generic form of the topological invariants (I_k and J_k) with respect to the conserved and on-shell $(\partial_\mu D^\mu C = D_\mu\partial^\mu\bar{C} = 0)$ nilpotent $\left(Q_b^2 = Q_d^2 = 0\right)$ BRST (Q_b) and co-BRST (Q_d) charges, on the 2D manifold, is

$$
I_k = \oint_{C_k} V_k \qquad J_k = \oint_{C_k} W_k \qquad (k = 0, 1, 2) \tag{2.8}
$$

[4] We follow here the notation and conventions of [30]. In fact, in its full glory, a nilpotent $(\delta^2_{(D)B} = 0)$ (co-)BRST transformation $(\delta_{(D)B})$ is equivalent to the product of an anti-commuting $(\eta C = -C\eta, \eta\bar{C} = -\bar{C}\eta)$ spacetime-independent parameter η and $(s_d)s_b$ (i.e. $\delta_{(D)B} = \eta s_{(d)b}$) where $s^2_{(d)b} = 0$.

where C_k are the k-dimensional homology cycles in the 2D manifold and V_k and W_k are the k-forms w.r.t. Q_b and Q_d, respectively[5]. These forms are [5, 6]

$$V_0 = -(\partial_\rho A^\rho) \cdot C - \frac{\mathrm{i}}{2}\bar{C} \cdot C \times C \qquad V_1 = [-(\partial_\rho A^\rho) \cdot A_\mu + \mathrm{i}C \cdot D_\mu \bar{C}]\, \mathrm{d}x^\mu$$
$$V_2 = \mathrm{i}[A_\mu \cdot D_\nu \bar{C} - \bar{C} \cdot D_\mu A_\nu]\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{2.9}$$

$$W_0 = E \cdot \bar{C} \qquad W_1 = [\bar{C} \cdot \varepsilon_{\mu\rho}\partial^\rho C - \mathrm{i}E \cdot A_\mu]\, \mathrm{d}x^\mu$$
$$W_2 = \mathrm{i}\left[\varepsilon_{\mu\rho}\partial^\rho C \cdot A_\nu + \tfrac{1}{2} C \cdot \varepsilon_{\mu\nu}(\partial_\rho A^\rho)\right] \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{2.10}$$

These topological invariants obey certain specific kinds of recursion relations [5, 6] which primarily shed some light on the connection between (co-)BRST transformations *and* operation of (co-)exterior derivatives on these invariants (cf (5.4)). Equations (2.5)–(2.10) establish the topological nature of the above self-interacting 2D non-Abelian gauge theory. For this theory, with off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries and corresponding conserved and off-shell nilpotent charges, a set of four topological invariants has been computed in [6]. In what follows hereafter, we shall concentrate, however, only on the Lagrangian density (2.1), its on-shell nilpotent symmetry generators (i.e. (co-)BRST charges) and corresponding topological invariants.

3. On-shell nilpotent BRST symmetry: superfield formulation

To provide the geometrical interpretation for the conserved and on-shell nilpotent BRST charge Q_b (cf (2.3)) as the translation generator in the framework of the superfield formulation [19–23], we first generalize the basic generic local field $\Psi(x) = (A_\mu(x), C(x), \bar{C}(x))$ of the Lagrangian density (2.1) to a chiral ($\partial_\theta V_s(x, \theta, \bar{\theta}) = 0$) supervector superfield $V_s = (B_\mu(x, \bar{\theta}), \Phi(x, \bar{\theta}), \bar{\Phi}(x, \bar{\theta}))$ defined on the (2 + 2)-dimensional supermanifold with the following super expansions along the Grassmannian direction $\bar{\theta}$ of the supermanifold:

$$\begin{aligned}
\left(B_\mu^a T^a\right)(x, \bar{\theta}) &= \left(A_\mu^a T^a\right)(x) + \bar{\theta}\left(R_\mu^a T^a\right)(x) \\
(\Phi^a T^a)(x, \bar{\theta}) &= (C^a T^a)(x) - \mathrm{i}\bar{\theta}(\mathcal{B}^a T^a)(x) \\
(\bar{\Phi}^a T^a)(x, \bar{\theta}) &= (\bar{C}^a T^a)(x) + \mathrm{i}\bar{\theta}(B^a T^a)(x)
\end{aligned} \tag{3.1}$$

where the signs have been chosen for later algebraic convenience only. Some of the salient and relevant points at this juncture are: (i) in general, the (2 + 2)-dimensional supermanifold is parametrized by the superspace coordinates $Z^M = (x^\mu, \theta, \bar{\theta})$ where x^μ ($\mu = 0, 1$) are the even spacetime coordinates and $\theta, \bar{\theta}$ are the odd Grassmannian variables ($\theta^2 = \bar{\theta}^2 = 0, \theta\bar{\theta} + \bar{\theta}\theta = 0$). However, here we choose only the chiral superfields which depend only on the superspace variables $Z^M = (x^\mu, \bar{\theta})$. (ii) The most general expansions for the even superfield B_μ and odd superfields Φ and $\bar{\Phi}$, along θ-, $\bar{\theta}$- and $\theta\bar{\theta}$-directions of the

[5] It will be noted that we have chosen a 2D Minkowskian manifold where metric has opposite signs in the diagonal entries. To be very precise, this manifold is not a compact manifold. To have the accurate meaning of the topological invariants, homology cycles, etc (and their connections with the notions in the algebraic geometry), one has to consider the Euclidean version of the 2D Minkowskian manifold which turns out to be the 2D closed Riemann surface. In this case, the 2D metric will have the same signs in the diagonal entries and $\mu, \nu, \rho, \ldots = 1, 2$. In fact, on these lines, a detailed analysis for the 2D (non-)Abelian gauge theories has been performed in [32]. For the sake of brevity, however, we shall continue with our Minkowskian notation, but we shall keep in mind this crucial point and decisive argument.

supermanifold, are [17, 21][6]

$$
\begin{aligned}
\left(B_\mu^a T^a\right)(x,\bar\theta) &= \left(A_\mu^a T^a\right)(x) + \bar\theta\left(R_\mu^a T^a\right)(x) + \theta\left(\bar R_\mu^a T^a\right)(x) + \mathrm{i}\theta\bar\theta\left(S_\mu^a T^a\right)(x)\\
(\Phi^a T^a)(x,\bar\theta) &= (C^a T^a)(x) - \mathrm{i}\bar\theta(\mathcal{B}^a T^a)(x) + \mathrm{i}\theta(\bar B^a T^a)(x) + \mathrm{i}\theta\bar\theta(s^a T^a)(x)\\
(\bar\Phi^a T^a)(x,\bar\theta) &= (\bar C^a T^a)(x) + \mathrm{i}\bar\theta(B^a T^a)(x) - \mathrm{i}\theta(\bar{\mathcal{B}}^a T^a)(x) + \mathrm{i}\theta\bar\theta(\bar s^a T^a)(x).
\end{aligned}
\tag{3.2}
$$

The chiral limit (i.e. $\theta \to 0$) of the above expansion has been taken into (3.1). (iii) The horizontality condition (see, e.g., [17, 21]) on (3.2) leads to the derivation of the off-shell nilpotent (anti-)BRST symmetries. We shall see later that the same condition on (3.1) yields the on-shell nilpotent BRST symmetry *alone*. (iv) The auxiliary fields in (3.1) are the fermionic (odd) fields R_μ and bosonic (even) fields B and $\mathcal{B}$. The corresponding fields in (3.2) are $R_\mu, \bar R_\mu, s, \bar s$ and $B, \bar B, \mathcal{B}, \bar{\mathcal{B}}, S_\mu$. (v) In expansions (3.1) and (3.2), all the local fields on the rhs are functions of the spacetime variables x^μ alone.

Now we invoke the horizontality condition ($\tilde F = \tilde{\mathrm{D}}\tilde A = \mathrm{D}A = F$) on the super curvature 2-form $\tilde F = \frac{1}{2}(\mathrm{d}Z^M \wedge \mathrm{d}Z^N)\tilde F_{MN}$ by exploiting the Maurer–Cartan equation, as

$$
\tilde F = \tilde{\mathrm{d}}\tilde A + \tilde A \wedge \tilde A \equiv \mathrm{d}A + A \wedge A = F \tag{3.3}
$$

where the super 1-form connection $\tilde A$ (in terms of the *chiral* superfields) and super exterior derivative $\tilde{\mathrm{d}}$ (in terms of the *chiral* superspace variables $(x^\mu, \bar\theta)$) are defined as

$$
\begin{aligned}
\tilde A &= \mathrm{d}Z^M \tilde A_M = \mathrm{d}x^\mu B_\mu(x,\bar\theta) + \mathrm{d}\bar\theta\,\Phi(x,\bar\theta)\\
\tilde{\mathrm{d}} &= \mathrm{d}Z^M \partial_M = \mathrm{d}x^\mu \partial_\mu + \mathrm{d}\bar\theta\,\partial_{\bar\theta}.
\end{aligned}
\tag{3.4}
$$

The above definitions lead to the following expressions:

$$
\begin{aligned}
\tilde{\mathrm{d}}\tilde A &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(\partial_\mu B_\nu) + (\mathrm{d}x^\mu \wedge \mathrm{d}\bar\theta)(\partial_\mu\Phi - \partial_{\bar\theta}B_\mu) - (\mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta)(\partial_{\bar\theta}\Phi)\\
\tilde A \wedge \tilde A &= (\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu)(B_\mu B_\nu) + (\mathrm{d}x^\mu \wedge \mathrm{d}\bar\theta)([B_\mu, \Phi]) - \tfrac{1}{2}(\mathrm{d}\bar\theta \wedge \mathrm{d}\bar\theta)(\{\Phi, \Phi\}).
\end{aligned}
\tag{3.5}
$$

Now the horizontality restrictions in (3.3) imply the following:

$$
\begin{aligned}
&\partial_\mu\Phi - \partial_{\bar\theta}B_\mu + [B_\mu, \Phi] = 0 \qquad \partial_{\bar\theta}\Phi + \tfrac{1}{2}\{\Phi, \Phi\} = 0\\
&\partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].
\end{aligned}
\tag{3.6}
$$

The first two relations in the above equation lead to the following expressions for the auxiliary fields in terms of the basic fields of the Lagrangian density (2.1):

$$
R_\mu(x) = D_\mu C(x) \qquad \mathcal{B}(x) = -\tfrac{\mathrm{i}}{2}(C \times C)(x) \qquad [\mathcal{B}(x), C(x)] = 0 \tag{3.7}
$$

and the lhs of the last relationship (with $\tilde F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$) yields

$$
\tilde F_{\mu\nu} = F_{\mu\nu} + \bar\theta(D_\mu R_\nu - D_\nu R_\mu) \equiv F_{\mu\nu} + \bar\theta F_{\mu\nu} \times C \tag{3.8}
$$

where we have used $D_\mu R_\nu = \partial_\mu R_\nu + [A_\mu, R_\nu]$, $R_\mu = D_\mu C$, $[D_\mu, D_\nu]C = F_{\mu\nu} \times C$. The total antisymmetric property of f^{abc} in a, b, c allows one to trivially check that the kinetic energy term of the Lagrangian density (2.1) remains invariant under transformation (3.8) $\left(\text{i.e. } -\frac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu} = -\frac{1}{4}\tilde F^{\mu\nu}\cdot\tilde F_{\mu\nu}\right)$. Thus, physically, the horizontality condition implies that the kinetic energy term of the Lagrangian density remains invariant (and equal to the square of the *ordinary* curvature tensor). In other words, the supersymmetric contribution coming from the $\bar\theta$-component of the super curvature tensor $\tilde F_{\mu\nu}$ *does not* lead to any additional changes to the usual kinetic energy term $\left(-\frac{1}{4}F^{\mu\nu}\cdot F_{\mu\nu}\right)$ which is defined on the ordinary 2D spacetime manifold. It is obvious from equation (3.7) that the horizontality restriction

[6] All the signs in the analogue of the super expansions (3.2) have been taken to be *only* positive in [17, 21]. We invoke here some negative signs for the sake of later algebraic convenience without changing the physical content of the theory.

(3.3) does not fix the auxiliary field $B(x) = (B^a T^a)(x)$ in terms of the basic fields of the Lagrangian density (2.1). However, it has been shown [5, 6] that the off-shell nilpotent BRST and co-BRST symmetries can be derived if we linearize the kinetic and gauge-fixing terms of (2.1) by invoking two auxiliary fields $\mathcal{B}$ and B in the following way:

$$\mathcal{L}_{\mathcal{B}} = \mathcal{B} \cdot E - \tfrac{1}{2}\mathcal{B} \cdot \mathcal{B} + (\partial_\rho A^\rho) \cdot B + \tfrac{1}{2} B \cdot B - \mathrm{i}\partial_\mu \bar{C} \cdot D^\mu C \tag{3.9}$$

which shows that $B = -(\partial_\rho A^\rho)$.[7] Thus, the super expansion in (3.1) can be re-expressed, in terms of the expressions for the auxiliary fields in (3.7) and $B = -(\partial_\rho A^\rho)$, as

$$\begin{aligned} B_\mu(x, \bar{\theta}) &= A_\mu(x) + \bar{\theta}(D_\mu C)(x) \equiv A_\mu(x) + \bar{\theta}(s_b A_\mu(x)) \\ \Phi(x, \bar{\theta}) &= C(x) - \tfrac{1}{2}\bar{\theta}(C \times C)(x) \equiv C(x) + \bar{\theta}(s_b C(x)) \\ \bar{\Phi}(x, \bar{\theta}) &= \bar{C}(x) - \mathrm{i}\bar{\theta}(\partial_\rho A^\rho)(x) \equiv \bar{C}(x) + \bar{\theta}(s_b \bar{C}(x)). \end{aligned} \tag{3.10}$$

With the above expansions as inputs, the on-shell nilpotent BRST symmetries in (2.2) can be concisely expressed in the language of superfields as

$$s_b B_\mu = \partial_\mu \Phi + (B_\mu \times \Phi) \qquad s_b \Phi = -\tfrac{1}{2}(\Phi \times \Phi) \qquad s_b \bar{\Phi} = -\mathrm{i}(\partial_\mu B^\mu). \tag{3.11}$$

In fact, in the above *three* transformations, the first yields $s_b A_\mu = D_\mu C$, $s_b C = -\frac{1}{2} C \times C$; the second produces $s_b C = -\frac{1}{2} C \times C$ and the third leads to $s_b \bar{C} = -\mathrm{i}(\partial_\mu A^\mu)$, $s_b(\partial_\mu A^\mu) = \partial_\mu D^\mu C$ in terms of the basic fields of (2.1). Comparing with (2.4) it is clear that

$$\mathrm{i}\frac{\partial}{\partial \bar{\theta}} B_\mu(x, \bar{\theta}) = [Q_b, A_\mu] \qquad \mathrm{i}\frac{\partial}{\partial \bar{\theta}} \Phi(x, \bar{\theta}) = \{Q_b, C\} \qquad \mathrm{i}\frac{\partial}{\partial \bar{\theta}} \bar{\Phi}(x, \bar{\theta}) = \{Q_b, \bar{C}\} \tag{3.12}$$

which shows that the conserved and on-shell nilpotent BRST charge Q_b (that generates the BRST transformations (2.2)) can be *geometrically* interpreted as the generator of translation $\left(\frac{\partial}{\partial \bar{\theta}}\right)$ along the Grassmannian direction $\bar{\theta}$ of the supermanifold. This clearly establishes the fact that the horizontality condition w.r.t. the super covariant derivative $\tilde{\mathrm{D}} = \tilde{\mathrm{d}} + \tilde{A}$ (in $\tilde{F} = \tilde{\mathrm{D}}\tilde{A} = \mathrm{D}A = F$) leads to the derivation of the on-shell nilpotent BRST symmetries for the non-Abelian gauge theory and it plays an important role in providing the geometrical interpretation for the BRST charge Q_b on the (2 + 2)-dimensional supermanifold.

4. On-shell nilpotent co-BRST symmetry: superfield approach

It is evident from our earlier discussions that $F = \mathrm{D}A = \mathrm{d}A + A \wedge A$ defines the 2-form curvature tensor on an ordinary compact manifold. The operation of an ordinary co-exterior derivative $\delta = -{*}\mathrm{d}{*}$ on the 1-form $A = \mathrm{d}x^\mu A^a_\mu T^a$ leads to the definition of the gauge-fixing term (i.e. $\delta A = \partial_\mu A^{\mu a} T^a$). Interestingly, the operation of the covariant co-exterior derivative $\Omega = -{*}\mathrm{D}{*}$ on the 1-form A leads to the same gauge-fixing term. This can be seen (with ${*}(\mathrm{d}x^\mu) = \varepsilon^{\mu\nu}(\mathrm{d}x_\nu)$, ${*}(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) = \varepsilon^{\mu\nu}$, ${*}A = \varepsilon^{\mu\nu}(\mathrm{d}x_\nu)A_\mu$) as follows:

$$\Omega A = -{*}(\mathrm{d} + A) * A = -{*}\, \mathrm{d}({*}A) - {*}(A \wedge {*}A) = \partial_\mu A^{\mu a} T^a \tag{4.1}$$

where the total antisymmetry property of the f^{abc} plays an important role in proving $A_\mu \times A^\mu = 0$. In the simplest way, this statement can be verified by noting that $\mathrm{D}_\mu A^\mu = \partial_\mu A^\mu + A_\mu \times A^\mu = \partial_\mu A^\mu$, which is, in essence, the reflection of (4.1).

Now we shall generalize the horizontality condition ($\tilde{\mathrm{D}}\tilde{A} = \mathrm{D}A$) of equation (3.3) (where (super) exterior derivatives ($\tilde{\mathrm{d}}$) d and (super) 1-forms ($\tilde{A}$) A play an important role) to the case

[7] For clarity, it is to be noted that the off-shell nilpotent BRST ($\tilde{s}_b$) and co-BRST ($\tilde{s}_d$) symmetry transformations: $\tilde{s}_b A_\mu = D_\mu C, \tilde{s}_b \bar{C} = \mathrm{i}B, \tilde{s}_b B = 0, \tilde{s}_b C = -\frac{1}{2} C \times C, \tilde{s}_b \mathcal{B} = \mathcal{B} \times C, \tilde{s}_b E = E \times C, \tilde{s}_b(\partial_\rho A^\rho) = \partial_\rho D^\rho C$ and $\tilde{s}_d A_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}, \tilde{s}_d C = -\mathrm{i}\mathcal{B}, \tilde{s}_d B = 0, \tilde{s}_d \bar{C} = 0, \tilde{s}_d \mathcal{B} = 0, \tilde{s}_d(\partial_\rho A^\rho) = 0, \tilde{s}_d E = D_\mu \partial^\mu \bar{C}, \tilde{s}_d(D_\mu \partial^\mu \bar{C}) = 0$ leave the Lagrangian density (3.9) invariant (up to a total derivative).

where the (super) co-exterior derivatives $(\tilde{\delta})\,\delta$ operate on (super) 1-forms $(\tilde{A})\,A$ to define a (super-)scalar. Thus, the analogue of the horizontality condition[8] is

$$\tilde{\delta}\tilde{A} = \delta A \qquad \delta A = (\partial_\mu A^\mu) \qquad \delta = -*\mathrm{d}* \qquad \tilde{\delta} = -\star\,\tilde{\mathrm{d}}\star \qquad A = \mathrm{d}x^\mu A_\mu \tag{4.2}$$

where, in the definition of $\tilde{\delta}\tilde{A} = -\star\,\tilde{\mathrm{d}}\star\tilde{A}$, we have to take into account

$$\tilde{\mathrm{d}} = \mathrm{d}x^\mu\partial_\mu + \mathrm{d}\bar{\theta}\partial_{\bar{\theta}} \qquad \star\,\tilde{A} = \varepsilon^{\mu\nu}(\mathrm{d}x_\nu)B_\mu(x,\bar{\theta}) + \mathrm{d}\bar{\theta}\bar{\Phi}(x,\bar{\theta}) \tag{4.3}$$

so that the operation of $\tilde{\mathrm{d}}$ on the 1-form $(\star\tilde{A})$ can exist in the chiral space. Here the Hodge duality $\star$ operation is defined on the $(2+2)$-dimensional supermanifold. In its most general form, this operation on the super differentials and their wedge products are

$$\begin{array}{lll}
\star\,(\mathrm{d}x^\mu) = \varepsilon^{\mu\nu}(\mathrm{d}x_\nu) & \star\,(\mathrm{d}\theta) = (\mathrm{d}\bar{\theta}) & \star\,(\mathrm{d}\bar{\theta}) = (\mathrm{d}\theta) \\
\star\,(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) = \varepsilon^{\mu\nu} & \star\,(\mathrm{d}x^\mu\wedge\mathrm{d}\theta) = \varepsilon^{\mu\theta} & \star\,(\mathrm{d}x^\mu\wedge\mathrm{d}\bar{\theta}) = \varepsilon^{\mu\bar{\theta}} \\
\star\,(\mathrm{d}\theta\wedge\mathrm{d}\theta) = s^{\theta\theta} & \star\,(\mathrm{d}\theta\wedge\mathrm{d}\bar{\theta}) = s^{\theta\bar{\theta}} & \star\,(\mathrm{d}\bar{\theta}\wedge\mathrm{d}\bar{\theta}) = s^{\bar{\theta}\bar{\theta}}
\end{array} \tag{4.4}$$

where $\varepsilon^{\mu\theta} = -\varepsilon^{\theta\mu}$, $\varepsilon^{\mu\bar{\theta}} = -\varepsilon^{\bar{\theta}\mu}$, $s^{\theta\bar{\theta}} = s^{\bar{\theta}\theta}$, etc. The choice of $(\star\tilde{A})$ in (4.3) is derived from the following general expression for the super 1-form $\tilde{A}$ and the application of the $\star$ operation (4.4) on it, in the $(2+2)$-dimensional supermanifold:

$$\begin{aligned}
\tilde{A}(x,\theta,\bar{\theta}) &= \mathrm{d}x^\mu B_\mu(x,\theta,\bar{\theta}) + \mathrm{d}\theta\bar{\Phi}(x,\theta,\bar{\theta}) + \mathrm{d}\bar{\theta}\Phi(x,\theta,\bar{\theta}) \\
\star\,\tilde{A}(x,\theta,\bar{\theta}) &= \varepsilon^{\mu\nu}(\mathrm{d}x_\nu)B_\mu(x,\theta,\bar{\theta}) + \mathrm{d}\bar{\theta}\bar{\Phi}(x,\theta,\bar{\theta}) + \mathrm{d}\theta\Phi(x,\theta,\bar{\theta}).
\end{aligned} \tag{4.5}$$

Taking the chiral limit $(\theta\to 0, \mathrm{d}\theta\to 0)$ of the above equation leads to the proof for the choice of $(\star\tilde{A})$ in (4.3). With the help of (4.4) and (4.3), the lhs of the dual horizontality condition (4.2) can explicitly be written as

$$\tilde{\delta}\tilde{A} = (\partial_\mu B^\mu) + s^{\bar{\theta}\bar{\theta}}(\partial_{\bar{\theta}}\bar{\Phi}) - \varepsilon^{\mu\bar{\theta}}(\partial_\mu\bar{\Phi} + \varepsilon_{\mu\nu}\partial_{\bar{\theta}}B^\nu). \tag{4.6}$$

Application of the requirement in (4.2) allows us to set the coefficients of $\varepsilon^{\mu\bar{\theta}}$ and $s^{\bar{\theta}\bar{\theta}}$ equal to zero. This restriction leads to the following relationships:

$$R_\mu(x) = -\varepsilon_{\mu\nu}\partial^\nu\bar{C}(x) \qquad B(x) = 0. \tag{4.7}$$

The other restriction, ensuing from (4.2), is $\partial_\mu B^\mu = \partial_\mu A^\mu$, which leads to $\partial_\mu R^\mu = 0$. It is evident that (4.7) automatically satisfies this condition. Physically, the dual horizontality condition amounts to the restriction that the ordinary gauge-fixing term $(\partial\cdot A)$ defined on the ordinary 2D spacetime manifold remains intact and unchanged. In other words, the supersymmetric contribution emerging from the coefficients $\varepsilon^{\mu\bar{\theta}}$, $s^{\bar{\theta}\bar{\theta}}$ in (4.6) does not alter the original value of the gauge-fixing term defined on the 2D ordinary spacetime manifold (i.e. $\delta A = (\partial\cdot A)$). It will be noted that the auxiliary field $\mathcal{B}(x)$ is not fixed by the dual horizontality condition in (4.2). However, our argument in the context of choice of the Lagrangian density (3.9) for the off-shell nilpotent BRST and co-BRST symmetries comes to our rescue as we have $\mathcal{B}(x) = E(x)$. Thus, expansion (3.1), with the results in (4.7), can be expressed as

$$\begin{aligned}
B_\mu(x,\bar{\theta}) &= A_\mu(x) - \bar{\theta}\varepsilon_{\mu\nu}\partial^\nu\bar{C}(x) \equiv A_\mu(x) + \bar{\theta}(s_d A_\mu(x)) \\
\Phi(x,\theta,\bar{\theta}) &= C(x) - \mathrm{i}\bar{\theta}E(x) \equiv C(x) + \bar{\theta}(s_d C(x)) \\
\bar{\Phi}(x,\theta,\bar{\theta}) &= \bar{C}(x) + \mathrm{i}\bar{\theta}(B(x)=0) \equiv \bar{C}(x) + \bar{\theta}(s_d\bar{C}(x)).
\end{aligned} \tag{4.8}$$

The above expansions (due to the dual horizontality condition in (4.2)) allow one to express the on-shell nilpotent co-BRST symmetry transformations of (2.2), in terms of the chiral superfields (4.8), as

$$s_d B_\mu = -\varepsilon_{\mu\nu}\partial^\nu\bar{\Phi} \qquad s_d\bar{\Phi} = 0 \qquad s_d\Phi = +\mathrm{i}\varepsilon^{\mu\nu}\left(\partial_\mu B_\nu + \tfrac{1}{2}B_\mu\times B_\nu\right) \tag{4.9}$$

[8] This condition has been christened the dual horizontality condition in [18, 26] because the (super) co-exterior derivatives $(\tilde{\delta})\,\delta$ are Hodge dual to (super) exterior derivatives $(\tilde{\mathrm{d}})\,\mathrm{d}$ on the (super) manifolds with (super) Hodge operations $(\star)\,*$ as defined in (4.2) and (4.4).

where the first transformation leads to $s_d A_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}$, $s_d \bar{C} = 0$, the second yields $s_d \bar{C} = 0$ and the third produces $s_d C = -\mathrm{i}E$, $s_d E = D_\mu \partial^\mu \bar{C}$ in the language of the basic fields of the Lagrangian density (2.1). Equation (4.8) establishes that the on-shell nilpotent co-BRST charge Q_d geometrically corresponds to the translation generator $\left(\frac{\partial}{\partial\bar\theta}\right)$ along the Grassmannian direction $\bar\theta$ of the supermanifold as

$$\frac{\partial}{\partial\bar\theta}\Sigma(x,\bar\theta) = -\mathrm{i}[\Lambda(x), Q_d]_\pm \qquad \Sigma = \Phi, \bar\Phi, B_\mu \qquad \Lambda = C, \bar{C}, A_\mu \tag{4.10}$$

where the bracket $[\ ,\ ]_\pm$ stands for the (anti-)commutator for the Σ (or corresponding Λ) being (fermionic) bosonic in nature and we have exploited the defining relationship (2.4).

5. Topological aspects: superfield formalism

We have derived in section 2 some of the key topological features in the Lagrangian formulation and shown that, modulo some total derivatives, the Lagrangian density (2.1) can be expressed as the sum of BRST and co-BRST anti-commutator (cf (2.5)). In the language of the chiral superfields, the same can be expressed, modulo total derivative $\partial_\mu X^\mu$, as

$$\mathcal{L}_B = -\frac{\mathrm{i}}{2}\frac{\partial}{\partial\bar\theta}\left(\left[\varepsilon^{\mu\nu}\left(\partial_\mu B_\nu + \tfrac{1}{2}B_\mu \times B_\nu\right)\cdot\Phi\right]|_{\text{co-BRST}} + [(\partial_\mu B^\mu)\cdot\bar\Phi]|_{\text{BRST}}\right) \tag{5.1}$$

where the subscripts BRST and co-BRST stand for the insertion of the chiral super expansions given in (3.10) and (4.8), respectively, and $X^\mu = \frac{\mathrm{i}}{2}(\bar{C}\cdot D^\mu C + \partial^\mu \bar{C}\cdot C)$. In the above computation, we have used

$$\left[\varepsilon^{\mu\nu}\left(\partial_\mu B_\nu + \tfrac{1}{2}B_\mu \times B_\nu\right)\cdot\Phi\right]|_{\text{co-BRST}} = -(E\cdot C + \bar\theta D_\mu\partial^\mu\bar{C}\cdot C) + \mathrm{i}\bar\theta E\cdot E.$$

Mathematically, the above Lagrangian density is nothing but the $\bar\theta$-component of the composite fields $(\partial_\mu B^\mu)\cdot\bar\Phi$ and $\varepsilon^{\mu\nu}\left(\partial_\mu B_\nu + \frac{1}{2}B_\mu \times B_\nu\right)\cdot\Phi$ when we substitute the chiral expansions (3.10) and (4.8) that have been obtained after the application of (dual) horizontality conditions. Incorporating the geometrical interpretation for the on-shell nilpotent (co-)BRST charges, it can be seen that the Lagrangian density (2.1) corresponds to the translation of some local (but composite) chiral superfields along the $\bar\theta$-direction of the $(2+2)$-dimensional supermanifold where the generators of the translation on the supermanifold are conserved and on-shell nilpotent BRST and co-BRST charges (Q_b and Q_d).

Let us now concentrate on the topological invariants of the theory. We can provide the geometrical origin for the zero-forms W_0 and V_0 of equations (2.10) and (2.9) which are (co-)BRST invariants. To this end, we note the following:

$$\begin{aligned}(\Phi\cdot\bar\Phi)|_{\text{BRST}} &= C\cdot\bar{C} + \mathrm{i}\bar\theta C\cdot(\partial_\mu A^\mu) - \tfrac{1}{2}\bar\theta(C\times C)\cdot\bar{C}\\ (\Phi\cdot\bar\Phi)|_{\text{co-BRST}} &= C\cdot\bar{C} - \mathrm{i}\bar\theta E\cdot\bar{C}.\end{aligned} \tag{5.2}$$

It is now obvious that zero-forms in (2.9) and (2.10) are as follows:

$$\mathrm{i}\frac{\partial}{\partial\bar\theta}(\Phi\cdot\bar\Phi)|_{\text{BRST}} = V_0 \qquad \mathrm{i}\frac{\partial}{\partial\bar\theta}(\Phi\cdot\bar\Phi)|_{\text{co-BRST}} = W_0. \tag{5.3}$$

Mathematically, it means that the on-shell ($\partial_\mu D^\mu C = 0$) BRST-invariant quantity V_0 is nothing but the $\bar\theta$-component of the local (but composite) chiral superfield $(\Phi\cdot\bar\Phi)$ when we substitute for them the super expansions (3.10) that are obtained after the imposition of the horizontality condition (3.3). In the language of the geometry on the supermanifold, V_0 is equivalent to a translation of the chiral superfield $(\Phi\cdot\bar\Phi)$ along the $\bar\theta$-direction which is generated by the on-shell nilpotent BRST charge Q_b. In a similar fashion, we can provide a geometrical interpretation of the on-shell ($D_\mu \partial^\mu \bar{C} = 0$) co-BRST-invariant zero-form W_0. The rest of the topological invariants ($V_k, W_k, k = 1, 2$) can be computed by the following recursion relations [5, 6] that characterize the topological nature of this theory:

$$s_b V_k = \mathrm{d} V_{k-1} \qquad s_d W_k = \delta W_{k-1} \qquad \mathrm{d} = \mathrm{d}x^\mu \partial_\mu \qquad \delta = \mathrm{i}\, \mathrm{d}x^\mu \varepsilon_{\mu\nu} \partial^\nu. \tag{5.4}$$

Now we provide the geometrical interpretation for the symmetric energy–momentum tensor $T^{(s)}_{\mu\nu}$ of the theory in the language of the translation on the (2+2)-dimensional supermanifold. In fact, it can be checked that the $T^{(s)}_{\mu\nu}$ of (2.6), modulo some total derivatives $X^{(s)}_{\mu\nu}$, can be expressed as

$$\begin{aligned} T^{(s)}_{\mu\nu} + X^{(s)}_{\mu\nu} &= \frac{\mathrm{i}}{2} \frac{\partial}{\partial \bar{\theta}} \left(\left[Y^{(s)}_{\mu\nu} \right] |_{\mathrm{BRST}} + \left[Z^{(s)}_{\mu\nu} \right] |_{\text{co-BRST}} \right) \\ Y^{(s)}_{\mu\nu} &= \partial_\mu \bar{\Phi} \cdot B_\nu + \partial_\nu \bar{\Phi} \cdot B_\mu + \eta_{\mu\nu} (\partial_\rho A^\rho) \cdot \bar{\Phi} \\ Z^{(s)}_{\mu\nu} &= \varepsilon_{\mu\rho} \partial_\nu \Phi \cdot B^\rho + \varepsilon_{\nu\rho} \partial_\mu \Phi \cdot B^\rho + \eta_{\mu\nu} \varepsilon^{\rho\sigma} \left(\partial_\rho B_\sigma + \tfrac{1}{2} B_\rho \times B_\sigma \right) \cdot \Phi \end{aligned} \tag{5.5}$$

where the explicit form of the total derivative term $X^{(s)}_{\mu\nu}$ is

$$\begin{aligned} X^{(s)}_{\mu\nu} = \frac{1}{2} \partial_\mu [(\partial_\rho A^\rho) \cdot A_\nu + E \cdot \varepsilon_{\nu\rho} A^\rho] + \frac{1}{2} \partial_\nu [(\partial_\rho A^\rho) \cdot A_\mu + E \cdot \varepsilon_{\mu\rho} A^\rho] \\ - \frac{\mathrm{i}}{2} \eta_{\mu\nu} \partial_\rho [\partial^\rho \bar{C} \cdot C + \bar{C} \cdot D^\rho C]. \end{aligned} \tag{5.6}$$

It is obvious from (5.5) that $T^{(s)}_{\mu\nu}$ geometrically corresponds to the translations of the local (but composite) chiral superfields $Y^{(s)}_{\mu\nu}$ and $Z^{(s)}_{\mu\nu}$ along the $\bar{\theta}$-direction of the (2 + 2)-dimensional supermanifold. These translations are generated by the on-shell nilpotent BRST and co-BRST charges Q_b and Q_d, respectively. Mathematically, the expression for $T^{(s)}_{\mu\nu}$, modulo some total derivatives, is nothing but the $\bar{\theta}$-component of the composite chiral superfields $Y^{(s)}_{\mu\nu}$ and $Z^{(s)}_{\mu\nu}$ where expansions (4.8) and (3.10) for the chiral superfields are taken into account. Of course, these expansions are obtained after the imposition of (dual) horizontality conditions which play a very significant role here.

6. Conclusions

In our present investigation, we have concentrated on the key topological properties of the 2D self-interacting non-Abelian gauge theory in the framework of the *chiral* superfield formulation. Our key observations are: (i) it is the existence of the novel on-shell nilpotent co-BRST symmetry (together with the familiar on-shell nilpotent BRST symmetry) that enables us to furnish a convincing proof for the topological nature of the 2D self-interacting non-Abelian gauge theory in the Lagrangian formulation because the Lagrangian density and symmetric energy–momentum tensor for the theory turn out to be the sum of BRST- and co-BRST-invariant parts (cf (2.5) and (2.7)). (ii) In the framework of the superfield formulation, this fact is reflected in the appearance of the Lagrangian density and the symmetric energy–momentum tensor which turn out to be the total derivative w.r.t. a Grassmannian variable $\bar{\theta}$ (cf (5.1) and (5.5)). Geometrically, this is equivalent to the translation of some local (but composite) chiral superfields along the $\bar{\theta}$-direction of the supermanifold. These translations are basically generated by the conserved and on-shell nilpotent (co-)BRST charges. (iii) Our claim is that whenever a Lagrangian density and corresponding symmetric energy–momentum tensor turn out to be a total derivative w.r.t. a Grassmannian variable, the theory is a topological field theory and it owes its origin to the (super) cohomological operators ($\tilde{\mathrm{d}}$) d and/or ($\tilde{\delta}$) δ. (iv) It is important for our whole discussion (in this paper) to derive the on-shell nilpotent (co-)BRST symmetries in the superfield formulation because the Lagrangian density (2.1) is endowed with *only* these symmetries and it does not respect anti-BRST and anti-co-BRST symmetries. (v) The choice of the *chiral* superfields and imposition of the (dual) horizontality conditions enable us to derive the on-shell nilpotent BRST and co-BRST symmetries. This feature is different from the earlier attempts to derive the off-shell nilpotent (anti-)BRST symmetries [17, 20–22] (and (anti-)co-BRST symmetries [17]) in the framework of the superfield formulation where the most general super expansions for the superfields were considered. (vi) In our present analyses, the Lagrangian density (2.1) and the corresponding symmetric energy–momentum tensor (2.6) play key roles. Thus, the geometrical understanding of these physically relevant and interesting quantities *might* turn out to play an important role in the context of 2D gravity where a non-trivial (spacetime-dependent) metric is chosen for the discussion of such gauge theories in the background of the curved spacetime (super) manifolds.

It is a well-known fact that the 2D Abelian as well as non-Abelian (Yang–Mills) gauge theories possess no physical degrees of freedom when they are defined on an ordinary 2D spacetime manifold without any non-trivial topology at the boundary. In other words, for all such manifolds, the fields of the theory are assumed to fall off rapidly at infinity. Thus, our present 2D gauge theory, according to the standard definition of a TFT on a flat spacetime manifold with a spacetime-independent metric (see, e.g., [1] for details), turns out to be a new type of TFT because the (co-)BRST symmetries of the theory gauge out the propagating degrees of freedom. In our chiral superfield formulation, this fact is reflected in (5.1) and (5.5) where we have been able to show that the Lagrangian density and symmetric energy–momentum tensor of the theory are total Grassmannian derivatives, modulo some total spacetime derivatives. These latter derivatives do not contribute anything substantial in our present discussion. This will not be the case, however, if this 2D gauge theory is defined, say, on a circle where the boundary terms will contribute. In fact, for this case, there will be physical degrees of freedom associated with the gauge field. Furthermore, it is clear that all the physical fields of the theory will not go to zero at the boundary. Consequently, this theory will *not* be a TFT. In mathematical terms, now the total derivatives $\partial_\mu X^\mu$ and $X^{(s)}_{\mu\nu}$ of (5.1) and (5.5) cannot be neglected. As a result, the Lagrangian density and symmetric energy–momentum tensor will *not* be able to be expressed as the sum of (co-) BRST anti-commutators (in contrast to what we have shown in (2.5) and (2.7)). In the language of our present chiral superfield formulation, we shall *not* be able to express the above quantities as a total derivative w.r.t. the Grassmannian variable.

It would be nice to generalize our work to 4D 2-form free Abelian gauge theory where the existence of (co-)BRST symmetries has been shown [10]. On face value, however, it appears that there will be difficulty in such a generalization because of the fact that there is a single physical degree of freedom associated with the gauge field of the theory. But, we feel, it is important to try such a generalization so that we can learn more about some aspects of the 2-form gauge theory which plays a significant role in the context of string theory and field theory. In fact, some steps have already been taken in this direction [33]. The superfield formulation of the 2D interacting (non-)Abelian gauge theories is another direction that can be pursued later. It might be interesting to follow the approach adopted in [32] to discuss the 2D free Abelian and self-interacting non-Abelian gauge theories on the 2D closed Riemann surface (i.e. the Euclidean version of the 2D Minkowski manifold) of genus 1 (and/or higher genus Riemann surfaces) and study the topological invariants of this theory. It would be gratifying to find their connection with the pertinent notions in the domain of algebraic geometry.

These are some of the issues that are under investigation and our results will be reported elsewhere.

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